Clifford symbols. If V is a vector space with a quadratic form Q then

$$Cl(V,Q) = \bigotimes V \Big/ (v\otimes v - Q(v))$$

is an associative algebra. It is called the Clifford alg.

If V is a G-equivariant vector bundle over X then one defines a <u>G-equivariant Clifford symbol</u> for V as a pair (S, c), where S is a G-equivariant $\mathbb{Z}/2\mathbb{Z}$-graded hermitean vector bundle on X and $c : V \longrightarrow \mathrm{End}_G^{sa}(S)^{odd}$ a bundle map such that $c(v)^2 = \|v\|^2$.

(sa = selfadjoint)

Remark. At every $x \in X$ c_x extends to an algebra map $c_x : Cl(V_x) \longrightarrow \mathrm{End}^{sa}_{\mathbb{C}}(S_x)^{odd}$

Example. Let X be a Riemannian G-manifold, $V = TX$, $S = \Lambda^*(TX \otimes_{\mathbb{R}} \mathbb{C})$, $c(v) = v \wedge (-) + \iota_{v^*}(-)$

$v^*(w) := g(v,w)$, ι_{v^*} contraction.

Example. $V = \mathbb{R}^2 \longrightarrow \{*\} = X$. It is a G-bundle for $G = \mathbb{T}$ via the representation

$$z = e^{i\varphi} \longmapsto \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

$S = S^+ \oplus S^-$, $S^\pm = \mathbb{C}$; with the G-action

$$z \longmapsto \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \in GL(2,\mathbb{C}),$$

$c : \mathbb{R}^2 \to \mathrm{End}(\mathbb{C}^2)$, $c(x,y) = \begin{pmatrix} 0 & x+iy \\ x-iy & 0 \end{pmatrix}$

Then c is equivariant

$$c(gv) = g \circ c(v) \circ g^{-1}.$$

A homotopical point of view on K-theory.

X locally compact G-space.

Definition. Let E be a $\mathbb{Z}/2\mathbb{Z}$-graded bundle s.that $E^{\pm}$ are G-equivariant vector bundles admitting a G-equivariant complement to a trivial v. bundle, and $\varphi : E^+ \to E^-$ is a G-equivariant v. bundle map, which is iso off a compact set.

1. (E, φ) is called <u>degenerate</u> if φ is iso everywhere.
2. (E', φ') and (E'', φ'') are <u>stably isomorphic</u> if they become isomorphic after adding a degenerate (E, φ).
3. (E_0, φ_0) and (E_1, φ_1) are <u>homotopic</u> if there is

(E, φ) on $X \times [0,1]$ whose restriction to $X \times \{0\}$ and $X \times \{1\}$ are stably isomorphic to (E_0, φ_0) and (E_1, φ_1) respectively.

Segal's L-theory. Denote by $L^0_G(X)$ the set of homotopy classes of (E, φ)'s.
This is an abelian group. Next, Segal defines

$$L^{-n}_G(X) := L^0_G(X \times \mathbb{R}^n).$$

Theorem. $L_G^{-n}(X) \cong K_G^{-n}(X)$.

(Sketch of)
Proof. Given (E,φ), first add a degenerate (E',φ') so that E^- becomes trivial. So E then extends to the one-point compactification X^+. Let $U \subset X$ be a G-invariant open subset s.t. $\overline{U}$ is compact, and φ outside $\overline{U}$ is an isomorphism.

Form the "clutching" bundle

$$\tilde{E} := E^+|_{\overline{U}} \cup_{\partial U} E^-|_{X \setminus U}.$$

This is a trivial vector bundle outside of $\overline{U}$ so it extends to a G-equivariant vector bundle on X^+.

$\Rightarrow$ $[\tilde{E}] \in K^0_G(X^+)$ and $[E^-] \in K^0_G(X^+)$

$\Rightarrow$ $[\tilde{E}] - [V^-] \in K^0_G(X)$. So we obtain a map of abelian groups

$$L^0_G(X) \longrightarrow K^0_G(X).$$

The inverse is given by

$$[E \xrightarrow{id} E] \longleftarrow\!\shortmid [E]. \qquad \blacksquare$$

Exercise 27. Let $X = \mathbb{R}^2$, $E^\pm = X \times \mathbb{C}$. Show that any $\tilde{E}$ is iso to a power of the Hopf line bundle H^* on $(\mathbb{R}^2)^+ \cong S^2 \cong \mathbb{CP}^1$.

Solution.

the homotopy class of the transition function $S^1 \longrightarrow GL(1,\mathbb{C}) \cong \mathbb{C}\setminus\{0\}$ determines uniquely the iso class of a line bundle. But $\pi_1(\mathbb{C}\setminus\{0\}) \cong \mathbb{Z}$ the number $n \in \mathbb{Z}$ corresponds to the transition function $z \longmapsto z^n$. $\square$

$n = 1 \Leftrightarrow$ Hopf line bund

Let X be compact, $V \xrightarrow{\pi} X$ be a G-equiv. v. bundle on X.
Every Clifford symbol (S, c) for V defines
a Segal representative

$$E^{\pm} := \pi^* S^{\pm}, \quad \varphi = c$$

$c(v)^2 = \|v\|^2 \Rightarrow \varphi$ invertible
outside the zero section of V.
This gives (E, φ) on V.

Definition. Let $V \xrightarrow{\pi} X$ be an even rank Euclidean G-bundle over X. It is <u>G-K-orientable</u> if it admits a G-Clifford symbol (S, c) such that

$\operatorname{rank}(S) = 2^{\operatorname{rank}(V)/2}$.

Theorem. If $\pi : V \to X$ is G-K-orientable, then the class of (E, φ) obtained from the Clifford symbol (S, c) yields a vector bundle $\pi^* S \to V$. This yields a class $\tau_V \in K_G^{\operatorname{rank}(V)}(V)$, called the <u>Thom class</u>. Then the map

$$K_G^*(X) \longrightarrow K_G^{* + \operatorname{rank}(V)}(V)$$

$$\alpha \longmapsto \pi^*(\alpha) \cdot \tau_V$$

is a $R(G)$-module isomorphism.

Application to Dirac operators.

X complete Riemannian G-manifold.

- (S, c) a Clifford for TX
- ∇ a connection on S, i.e.

  i) G-equivariant map $\nabla_{gv}(gs) = g\nabla_v s$

  ii) compatible with the Levi-Civita connection

$$\nabla_v(c(w)s) = c(\nabla_v^{LC} w)s + c(w)\nabla_v s$$

  iii) Compatible with the Hermitian structure on S

$$\langle \nabla_v s, s' \rangle + \langle s, \nabla_v s' \rangle = v\langle s, s' \rangle .$$

Definition. Define a linear map D as the composition

$$\begin{tikzcd} \Gamma_c(S) \arrow[r, dashed, "D"] \arrow[d, "\nabla"'] & \Gamma_c(S) \\ \Gamma_c(T^*X \otimes S) \arrow[r, "\cong"] & \Gamma_c(TX \otimes S). \arrow[u, "\sqrt{-1}\, c"'] \end{tikzcd}$$

This is a G-equivariant operator on $\Gamma_c(S)$.

Exercise 28. Let (e_i) be a local orthonormal frame of TX. Find D in terms of ∇ and (e_i).

Solution. $Ds = \sqrt{-1} \sum_i c(e_i) \nabla_{e_i} s$. $\blacksquare$

Excercise 29. X Riemannian manifold, $S = \Lambda^*(TX \otimes \mathbb{C})$. with the Levi-Civita connection.

(non G-K-orientable case, since

$$\operatorname{rank}(S) = 2^{\dim X} \neq 2^{\operatorname{rank}(TX)/2} = 2^{\dim X/2})$$

$c : TX \longrightarrow \Lambda^*(TX \otimes \mathbb{C})$, $\quad c(v) = v \wedge (-) + \iota_{v^*}(-)$.

Show that then $D = d + d^*$, where d is the de Rham differential, is G-invariant with respect to $D = \mathrm{Isom}(X)$.

Exercise 30. The same for $S = \Lambda^*(T^*X \otimes \mathbb{C})$, $4 \mid \dim X$, where $S^{\pm}$ eigenbundles for the Hodge star $*$, $G = \mathrm{Iso}(X)^+$ (orientation preserving).

Exercise 31. The same for $S = \Lambda^* T^{(0,1)} X$, where X a Kähler manifold. Show that $\mathrm{rank}_{\mathbb{C}} S = 2^{\dim_{\mathbb{R}} X/2}$ (i.e. it is K-orientable case), $c(v) = v \wedge (-) + \iota_{v^*}(-)$ and $D = \bar\partial + \bar\partial^*$, where $\bar\partial$ is the Dolbeault operator; is equivariant with respect to any group G which acts by biholomorphic isometries.

Example. $X = SL(n,\mathbb{C})/B$, B Borel subgroup of upper triangular matrices. Then

$$X \cong SU(n)/T$$

where T is the subgroup of diagonal matrices. Then $G = SU(n)$ acts by holomorphic automorphisms of X.

Theorem. Let X be a complete Riemannian G-manifold, and (S, c) a G-Clifford symbol for TX, and let D be a Dirac operator. Then D has a unique selfadjoint extension $\bar{D}$ to $L^2(S)$.

In particular, functional calculus gives a $*$-homomorphism

$$\begin{aligned} C_b(\mathbb{R}) &\longrightarrow B(L^2(S)) \\ f &\longmapsto f(\bar{D}). \end{aligned}$$

Moreover, if X is compact, then $Sp(\bar{D}) \subseteq \mathbb{R}$ is discrete, all eigenvalues of $\bar{D}$ have finite dimensional eigenspaces that consist of smooth sections of S. $\blacksquare$

Remark. The n-th eigenvalue of $|D|$ is $\sim n^{1/\dim X}$.

In particular $f(\bar{D})$ is compact if $f \in C_0(\mathbb{R})$.

Moreover $\bar{D}$ is <u>Fredholm</u>, i.e. $\ker(\bar{D}^+)$ and $\ker(\bar{D}^-)$ are finite dimensional and G-invariant in $L^2(S)$.

(here $\bar{D}^\pm = \bar{D}|_{L^2(S^\pm)}$).

Definition (equivariant index)

$$\mathrm{ind}_G(D) := [\ker \bar{D}^+] - [\ker \bar{D}^-] \in R(G).$$